The Mathematics of Truth

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Laws of Truth

From Gottlob Frege's *Thoughts*:

> *Just as 'beautiful' points the ways for aesthetics and 'good' for ethics, so do words like 'true' for logic. All sciences have truth as their goal; but logic is also concerns with it in a quite different way: logic has much the same relation to truth as physics has to weight or heat. To discover truths is the task of all sciences; it falls to logic to discern the laws of truth.*

The Island of Knights and Knaves

- A certain series of puzzles, due to the logician Raymond Smullyan, highlight aspects of the logic of truth by considering various scenarios on the *Island of Knights and Knaves*.
- From *Satan, Cantor, and Infinity*:

> WITH A TWINGE of apprehension such as he had never felt before, an anthropologist named Abercrombie stepped onto the Island of Knights and Knaves. He knew that this island was populated by most perplexing people: knights, who make only true statements, and knaves, who make only false ones. "How," Abercrombie wondered, "am I ever to learn anything about this island if I can't tell who is lying and who is telling the truth?"

Puzzle 1

Abercrombie knew that before he could find out anything he would have to make one friend, someone whom he could always trust to tell him the truth. So when he came upon the first group of natives, three people, presumably named Arthur, Bernard, and Charles, Abercrombie thought to himself, "This is my chance to find a knight for myself." Abercrombie first asked Arthur, "Are Bernard and Charles both knights?" Arthur replied, "Yes." Abercrombie then asked: "Is Bernard a knight?" To his great surprise, Arthur answered: "No." Is Charles a knight or a knave?

Solution

- A is either a knight or a knave and A says the following two things:
  1. B and C are both knights.
  2. B is not a knight.
- Can these both be true?
- No! So they must both be false (because A must be a knave).
- This means:
  1. At least one of B and C is a knave.
  2. B is a knight.
- This means C is a knave.

Puzzle 2

- Abercrombie is told that, to discover the secrets of the Isle of Knights and Knaves, he must meet the Island Sorcerer. But in order to get an audience with the Sorcerer, he must identify the sorcerer's apprentice. He is told that the apprentice is one of three people in a certain room. He enters and asks who is the apprentice and receives the following replies:

"I am," replied one.

"I am the Sorcerer's Apprentice!" cried a second.

But the third remained silent.

"Can *you* tell me anything?" Abercrombie asked.

"It's funny," answered the third one with a sly smile. "At most, only one of the three of us ever tells the truth!"

Can it be deduced which of the three is the Sorcerer's Apprentice?

Solution

- The third speaker is either a knight or a knave and said
  - At most one of us tells the truth.
- What does it mean if this is false?
- If the third speaker's statement is false, then there are at least two of them that tell the truth.
- But, since the third speaker's statement was false, it must be that the first two speakers told the truth.
- But that is impossible because they can't *both* be the sorcerer's apprentice.
- So the third speaker must have told the truth (and must be the one truth-teller in the bunch).
- So the third speaker is the sorcerer's apprentice.

Exercise for the bored ones

38. Edward or Edwin?

This time you come across just one inhabitant lazily lying in the sun. You remember that his first name is either Edwin or Edward, but you cannot remember which. So you ask him his first name and he answers "Edward."

What is his first name?

Self-ascription

- Now suppose on the Island of Knights and Knaves, some individual of unknown identity says to you

  I am a knight.

  Can you determine whether this person is a knight or a knave?
- Suppose instead someone says

  I am a knave.

  What can you say about what kind of person they are?
- This sentence can't be said on the island: a knight wouldn't say it because it would be a lie, a knave wouldn't say it because it would be true.

The Liar Paradox

- Off the Island of Knights and Knaves, back in the real world, we can still consider a sentence like
  This sentence is false.
- Is it true or false?
- Our inability to say either that this sentence is true or false without arriving at contradiction is known as *The Liar Paradox*.

The Strengthened Liar Paradox

- What's the big deal? It isn't such a shocking idea to think that maybe some sentences in English are simply neither true nor false.
- But consider the following modified sentence:

  This sentence is either false or neither true nor false.
- This sentence can't be true. It can't be false. But it also can't be neither true nor false. This is the *Strengthened Liar Paradox*

Tarski's Theorem

- Tarski's Theorem on the Undefininability of Truth uses the Liar Paradox to show that we can't define the truths of arithmetic.
- Here's a more formal version. Suppose we have a list $P_1, P_2, P_3, \ldots$ of basic principles of arithmetic concerning $+, \times, -, 0, 1$ and a symbol T for truth, closed under deductive consequences. Suppose we have a coding of these principles, so $\ulcorner P_i \urcorner$ is some number that represents this proposition, in a sensible way. If we add principles that say

  $$P \leftrightarrow T(\ulcorner P \urcorner)$$

  for every possible formal expression P, the resulting set of principles is *inconsistent*.
- This means it is not possible, given a sensible coding of the statements of arithmetic, to define, within arithmetic, which ones are the true ones.

Self-reference

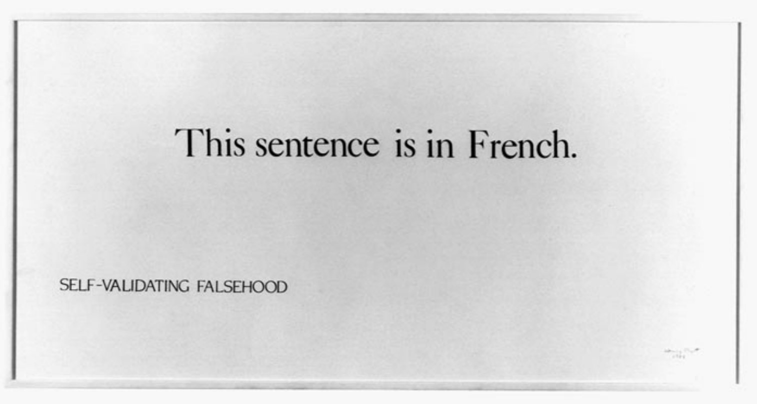

Figure: Is self-reference to blame?

Yablo's Paradox

- Instead of considering a *single* paradoxical statement, we will rather consider an infinite list of statements P_1, P_2, P_3, ...
- For each $i = 1, 2, 3, \ldots$, the statement P_i is the following assertion:

  For all $j > i, P_j$ is false.

- Is there a way to determine whether the statements in this list are true or false?

Yablo's Paradox

- For each $i = 1, 2, 3, \ldots$, the statement P_i is the following assertion:

  For all $j > i$, P_j is false.

- **Case 1**: There is some number i such that P_j is false for every $j > i$.
- **Case 2**: For every i, there is some $j \geq i$ such that P_j is true.

Yablo's Paradox

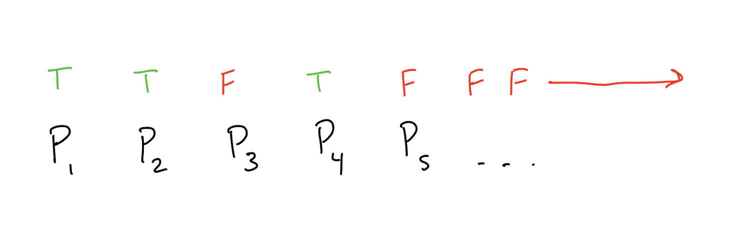

Figure: **Case 1**: There is some number i such that P_j is false for every $j > i$.

Yablo's Paradox

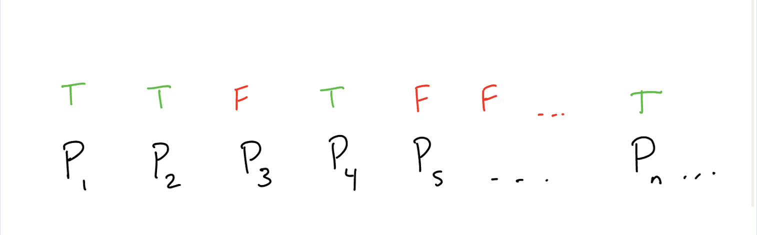

Figure: **Case 2**: For every i, there is some $j \geq i$ such that P_j is true.

Yablo's Paradox

- For each $i = 1, 2, 3, \ldots$, the statement P_i is the following assertion:

$$\text{For all } j > i, P_j \text{ is false.}$$

- **Case 1**: There is some number i such that P_j is false for every $j > i$.
- **Case 2**: For every i, there is some $j \geq i$ such that P_j is true.

Yablo's Paradox: Case 1

- **Case 1**: There is some number i such that P_j is false for every $j > i$.
- So, in particular, P_{i+1} is false.
- Also P_j is false for every $j > i + 1$.
- So P_{i+1} is true, because P_{i+1} simply said that P_j is false for every $j > i + 1$.
- Contradiction.

Yablo's Paradox: Case 2

- **Case 2**: For every i, there is some $j \geq i$ such that P_j is true.
- So suppose n is a number such that P_n is true.
- Since we're in Case 2, there is some $m > n$ such that P_m is true too.
- But P_n says that P_i is false for *every* $i > n$. So, in particular, P_m is false.
- Contradiction.

Yablo's Paradox

- Yablo's paradox resembles the Liar Paradox, in that any way of assigning true or false to the propositions in question seems to result in contradiction.
- But *unlike* the Liar Paradox there is no obvious self-reference.

Thanks!

Solution to the exercise for the bored ones

38.

I feel entitled, occasionally, to a little horseplay. The vital clue I gave you was that the man was lazily lying in the sun. From this it follows that he was lying in the sun. From this it follows that he was *lying*, hence he is a knave. So his name is Edwin.