Binarity, Treelessness, and Generic Stability

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This is joint work with Itay Kaplan and Pierre Simon

Generalities

- Here's one way to do model theory: come up with a combinatorial restriction on definability in a theory, and ask what structural/non-structural consequences it has.
- Some paradigms:
  - **Binarity**: Ask that the theory eliminate quantifiers in a relational language in which every symbol has arity at most 2.
  - **Classification-theoretic dividing lines**: Ask that the theory omit a certain pattern of consistency and/or inconsistency for partitioned formulas.
  - **Indiscernible collapse**: Ask that every generalized indiscernible of some kind is automatically a generalized indiscernible of some other kind.

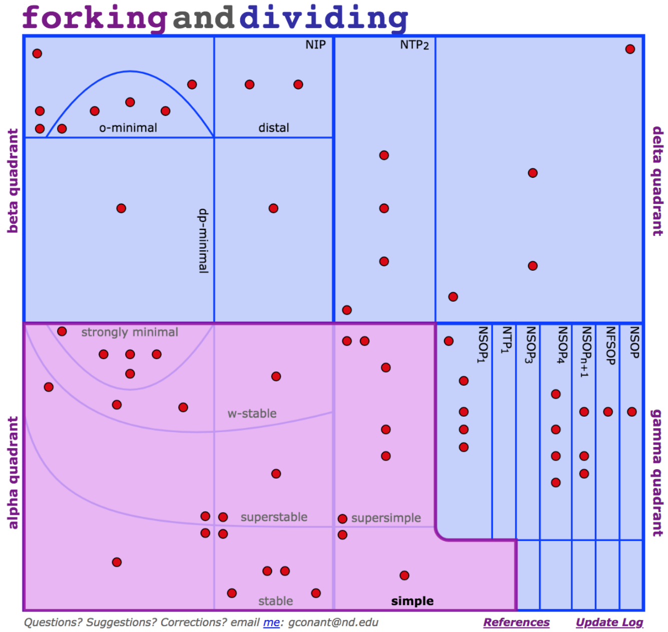
Binarity

- A theory T is called *binary* if, whenever given tuples $a = (a_0, \ldots, a_{n-1})$ and $b = (b_0, \ldots, b_{n-1})$, we have $a \equiv b$ if and only if, for all $i < j$, $a_i a_j \equiv b_i b_j$.
- When T is homogeneous ($\aleph_0$-categorical and eliminates quantifiers in a finite relational language), this is equivalent to having only unary and binary relation symbols in the language.
- **Examples**: DLO, the Fraïssé limit of finite equivalence relations, the random graph
- **Non-examples**: Dense $\wedge$-trees, random 3-hypergraph.

Binarity

- **Question**: What are the binary homogeneous structures?
- Since the mid-70s, complete classifications have been given for several classes of binary homogeneous structures:
  - Partial orders (Schmerl)
  - Graphs (Lachlan-Woodrow)
  - Directed graphs (Cherlin)
  - Tournaments (Lachlan)
  - Colored multi-partite graphs (Lockett, Truss)
  - ...

Classification-theoretic dividing lines



Generalized indiscernibles

- Suppose I is an L'-structure and $(a_i)_{i \in I}$ is a collection of tuples in the monster $\mathbb{M}$. We say $(a_i)_{i \in I}$ is an *I-indexed indiscernible* if, given any tuples $\overline{\eta} = (\eta_0, \ldots, \eta_{n-1})$ and $\overline{\nu} = (\nu_0, \ldots, \nu_{n-1})$ from I, we have

$$\mathrm{qftp}_{L'}(\overline{\eta}) = \mathrm{qftp}_{L'}(\overline{\nu}) \implies (a_{\eta_0}, \ldots, a_{\eta_{n-1}}) \equiv (a_{\nu_0}, \ldots, a_{\nu_{n-1}}).$$

- **Examples**:
  - If $I = (I, <)$ is an infinite linear order, then I-indexed indiscernibles are just indiscernible sequences.
  - If I is an infinite set in the language of equality, then I-indexed indiscernibles are indiscernible *sets*.

Indiscernible collapse

- **Stability**: A theory T is stable if and only if every indiscernible sequence is an indiscernible set (Shelah)
- **NIP**: A theory T is NIP if and only if every random ordered graph indiscernible is an indiscernible sequence (Scow)
- n-**dependence**: A theory T is n-dependent if and only if every random ordered $(n+1)$-ary hypergraph indiscernible is an indiscernible sequence (Chernikov-Palacín-Takeuchi)

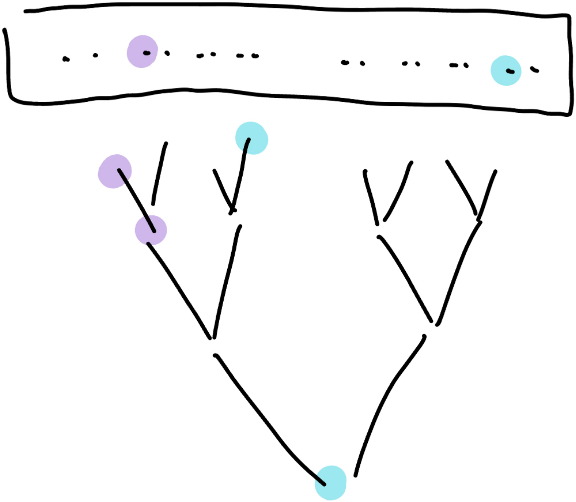
Binarity and Classification Theory

- From a model-theoretic point of view, it is more natural to ask how binarity interacts with classification-theoretic dividing lines.
- For stable structures this was done in the 80s (Lachlan-Shelah, Lachlan), giving a classification* of all stable homogeneous structures.
- For simple structures, classification results have been considered much more recently (Aranda-Lopez, Koponen), leading to a satisfying classification*.

Treetop indiscernibles

- Let $L_{0,P} = \{\trianglelefteq, \wedge, <_{lex}, P\}$ and consider $\omega^{\leq\omega}$ as an $L_{0,P}$ with the following interpretations:
  - $\trianglelefteq$ = tree partial order
  - $<_{lex}$ = lexicographic order
  - $\wedge$ = binary meet function
  - $P = \omega^\omega$, the leaves of the tree.
- We say an $\omega^{\leq\omega}$-indexed indiscernible (with $\omega^{\leq\omega}$ considered as an $L_{0,P}$-structure $(a_\eta)_{\eta\in\omega^{\leq\omega}}$ is a *treetop indiscernible*.
- This extends naturally to $(a_\eta)_{\eta\in\mathcal{T}}$ for any $L_{0,P}$-structure $\mathcal{T}$ with $\mathrm{Age}(\mathcal{T}) = \mathrm{Age}(\omega^{\leq\omega})$.

Treetop indiscernibles



Treeless theories

Definition

A theory T is called *treeless* if, whenever $(a_\eta)_{\eta \in \mathcal{T}}$ is a treetop indiscernible, $(a_\eta)_{\eta \in P(\mathcal{T})}$ is an indiscernible sequence (with $P(\mathcal{T})$ viewed as a dense linear order under $<_{lex}$).

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- Binary theories are treeless.
- Stable theories are treeless.
- If T' is interpretable in a treeless T, then T' is treeless.

Generic Stability

- We say that a partial type p is Ind-definable over A if for every $\varphi(x;y)$, the set $\{b : \varphi(x;b) \in \pi\}$ is Ind-definable over A (i.e., is a union of A-definable sets).
- Suppose π is a global partial type. We say π is *generically stable* over A if π is Ind-definable over A if: given any $\varphi(x;b) \in \pi$ and sequence $(a_i)_{i<\omega}$ with $a_k \models \pi|_{Aa_{<k}}$ for all k, we have

$$\models \varphi(a_k, b)$$

for all but finitely many k.

Generic stability

- If $\pi(x)$ and $\lambda(x)$ are global partial types, generically stable over A, then if $\pi(x)|_A \cup \lambda(x)|_A$ is consistent then $\pi(x) \cup \lambda(x)$ is consistent. Hence every $p \in S(A)$ extends to a *maximal* global generically stable partial type.
- Gives a notion of independence: $a \mathop{\smile\!\!\!\!\!\!\;\mid}_A^{\pi} b$ if $b \models \pi|_{Aa}$ for π the maximal generically stable extension of $\mathrm{tp}(b/A)$.

Generic stability

Theorem

If T is treeless then $\mathop{\smile\!\!\!\!\!|}^{\pi}$ is symmetric and satisfies base monotonicity—that is,

$$a \mathop{\smile\!\!\!\!\!|}^{\pi}_{A} bc \implies a \mathop{\smile\!\!\!\!\!|}^{\pi}_{Ab} c.$$

We will see that this has consequences for classification-theoretic dividing lines.

NSOP_1

- Recently, the class of NSOP_1 theories, which properly contains the simple theories, has been intensively studied. There is a structure theory for NSOP_1 theories completely parallel to that for simple theories, with the notion of *Kim-independence* playing the role that forking-independence plays for simple theories.
- There are lots of interesting examples:
  - **Combinatorics**: Generic structures (Kruckman-R.), generic projective planes (Conant-Kruckman), Steiner triple systems (Barbina-Casanovas), classical geometries over algebraically closed or pseudo-finite fields (Chernikov-R.)
  - **Algebra**: PAC fields with free Galois group (Kaplan-R.), Frobenius fields (Kaplan-R.), existentially closed G-fields for G virtually free* (Beyarslan-Kowalski-R.), Abelian varieties with a generic subgroup (d'Elbee), existentially closed exponential fields (Haykazyan-Kirby), Hilbert spaces with generic subset (Berenstein-Hyttinen-Villaveces)
- **Question**: What do binary homogeneous NSOP_1 structures look like?

T^*_{feq}

- The canonical example of a $NSOP_1$ non-simple homogeneous structure is T^*_{feq}, the generic theory of parameterized equivalence relations.
- More precisely, we let L_{feq} be the language with two sorts O (for 'objects') and P (for 'parameters'), as well as a ternary relation $E_x(y, z) \subseteq P \times O^2$. The class $\mathbb{K}_{feq}$ is the class of finite L_{feq}-structures A where, for all $p \in P(A)$, E_p is an equivalence relation on $O(A)$ is a Fraíssé class. T^*_{feq} is the theory of the Fraíssé limit.

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- More generally, if M is a homogeneous $NSOP_1$ structure, one can form its 'parameterized version' which will again be homogeneous $NSOP_1$ (Chernikov-R.)
- T^*_{feq} (and the non-simple $NSOP_1$ parametrized structures) are all (at least) *ternary*. Is there a binary $NSOP_1$ homogeneous structure?

Theorem

Theorem

If T is treeless and $NSOP_1$ then T is simple.

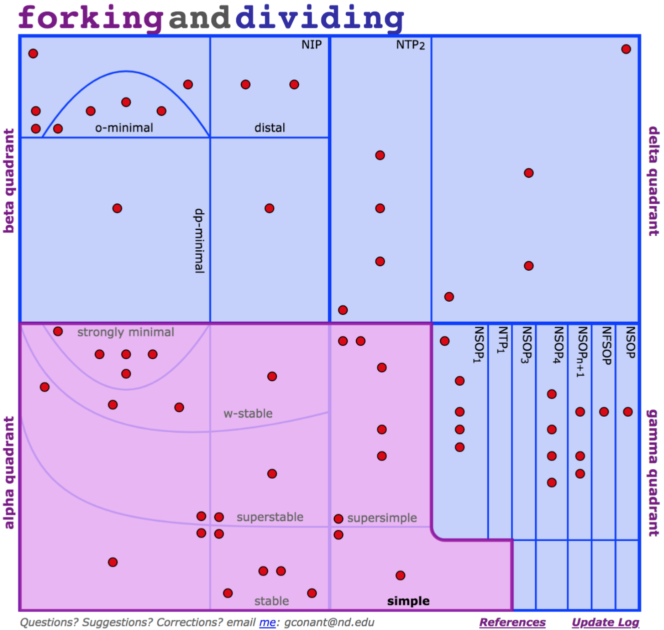
The Proof

- If T is NSOP$_1$, then $\mathop{\smile\hskip-0.9em\vert\hskip0.5em}^K = \mathop{\smile\hskip-0.9em\vert\hskip0.5em}^\pi$ over models (Kim's lemma for $\mathop{\smile\hskip-0.9em\vert\hskip0.5em}^K$).
- A theory is simple if and only if $\mathop{\smile\hskip-0.9em\vert\hskip0.5em}^K$ satisfies base monotonicity, that is, if $M \prec N \models T$ and $a \mathop{\smile\hskip-0.9em\vert\hskip0.5em}^K_M Nb$ then $a \mathop{\smile\hskip-0.9em\vert\hskip0.5em}^K_N b$ (Kaplan-R).
- Treelessness implies base monotonicity.

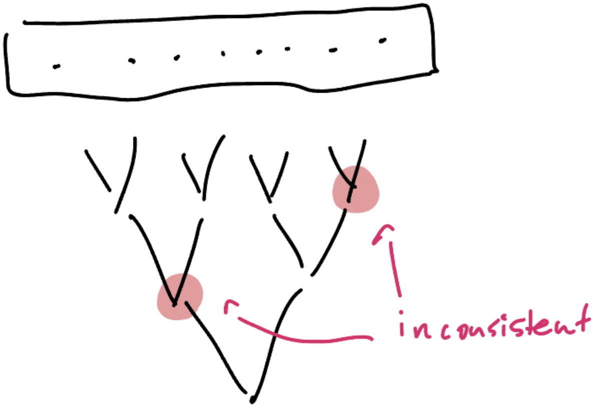
NSOP$_n$

- T has SOP$_2$ (=TP$_1$) if there is some formula $\varphi(x; y)$ and a tree of tuples $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that:
  - (Paths are consistent) For all $\eta \in \omega^\omega$, $\{\varphi(x; a_{\eta|k}) : k < \omega\}$ is consistent.
  - (Incomparables are inconsistent) For all $\eta \perp \nu \in \omega^{<\omega}$, $\{\varphi(x; a_\eta), \varphi(x; a_\nu)\}$ is inconsistent.
- T has SOP$_n$ for $n \geq 3$ if there is some formula $\varphi(x; y)$ and some indiscernible sequence $(a_i)_{i<\omega}$ such that:
  - $\models \varphi(a_i, a_j)$ if and only if $i < j$.
  - $\{\varphi(x_0, x_1), \varphi(x_1, x_2), \ldots, \varphi(x_{n-2}, x_{n-1}), \varphi(x_{n-1}, x_0)\}$ is inconsistent.
- For any n, T is said to be NSOP$_n$ if it does not have SOP$_n$.

The Map


forking and dividing
NIP
NTP2
o-minimal
distal
dp-minimal
beta quadrant
delta quadrant
strongly minimal
w-stable
superstable
supersimple
stable
simple
alpha quadrant
gamma quadrant
NSOP1
NTP1
NSOP3
NSOP4
NSOPn+1
NFSOP
NSOP
Questions? Suggestions? Corrections? email me: gconant@nd.edu
References
Update Log

Picture of SOP_2



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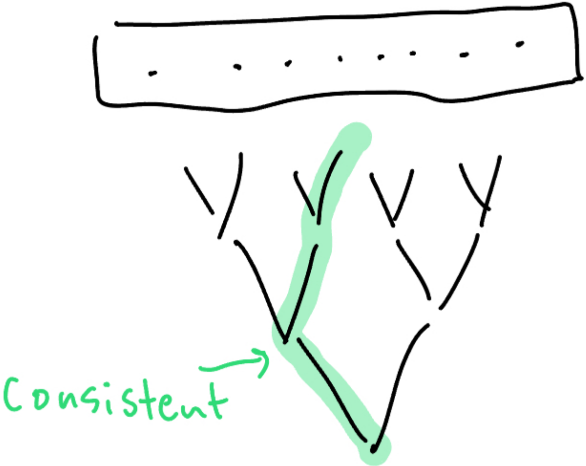


NSOP$_n$

- We have the following implications:

$$NSOP_1 \implies NSOP_2 \implies NSOP_n \implies NSOP_{n+1}$$

  for all $n \geq 3$.
- For each $n \geq 3$, there are (binary homogeneous) structures which are SOP$_n$ and NSOP$_{n+1}$.
- It ~~is~~ was open, in general, if NSOP$_1$, NSOP$_2$, and NSOP$_3$ are distinct.
- **Question**: What about when restricted to binary theories?

Theorem

Theorem

A treeless $NSOP_3$ theory with trivial indiscernibility is $NSOP_2$.

Definition

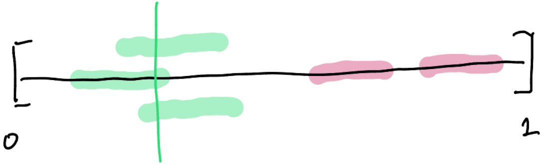
T is said to have *trivial indiscernibility* if whenever I is a-indiscernible and b-indiscernible, it is ab-indiscernible. It holds, e.g., in binary theories.

The proof

- The proof uses the following characterization of SOP_3: T has SOP_3 if and only if there is a formula $\varphi(x; y)$ and a collection of tuples $(a_I)_{I\in\mathcal{I}}$, where $\mathcal{I} = \{[a, b] \subseteq \mathbb{R} : 0 \leq a < b \leq 1\}$ such that, for all $J \subseteq \mathcal{I}$,

$$\{\varphi(x; a_I) : I \in J\} \text{ is consistent} \iff \bigcap J \neq \emptyset.$$

SOP_3



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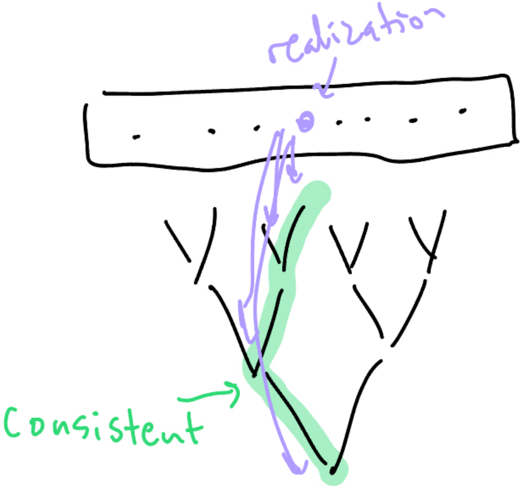
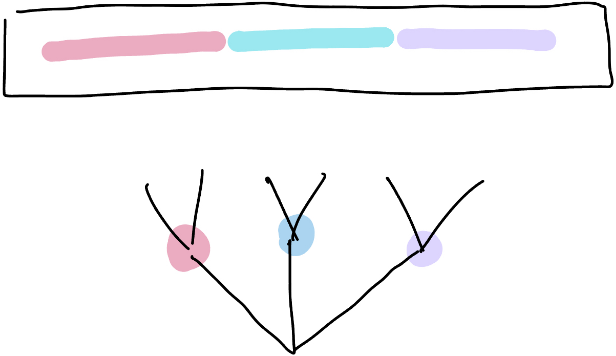
$$\{\varphi(x; a_I) : I \in J\} \text{ is consistent } \iff \bigcap J \neq \emptyset.$$

- We also choose a witness $(a_\eta)_{\eta \in \omega^{<\omega}}$ to SOP_2 such that $(a_\eta)_{\eta \in \omega^{\leq \omega}}$ forms a treetop indiscernible where each leaf realizes the path-type below it.

Treetop witness



Finding witnesses



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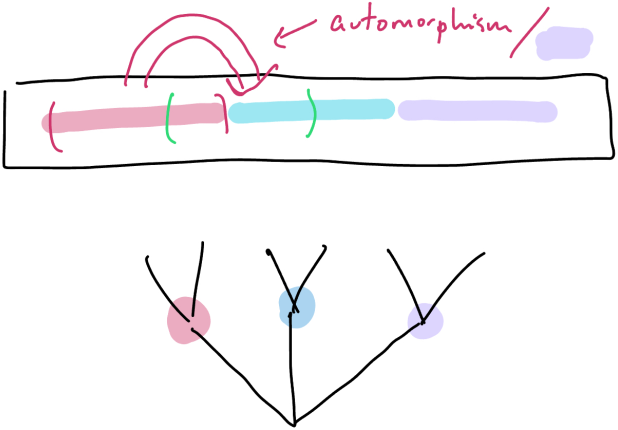

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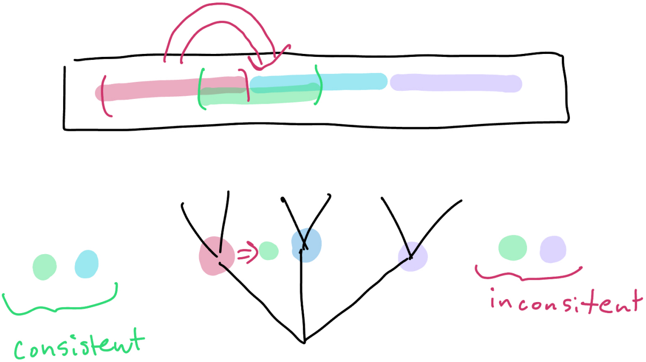

The proof

- The proof uses the following characterization of SOP_3: T has SOP_3 if and only if there is a formula $\varphi(x; y)$ and a collection of tuples $(a_I)_{I \in \mathcal{I}}$, where $\mathcal{I} = \{[a, b] \subseteq \mathbb{R} : 0 \leq a < b \leq 1\}$ such that, for all $J \subseteq \mathcal{I}$,

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- We also choose a witness $(a_\eta)_{\eta \in \omega^{<\omega}}$ to SOP_2 such that $(a_\eta)_{\eta \in \omega^{\leq\omega}}$ forms a treetop indiscernible where each leaf realizes the path-type below it.
- Treelessness lets us find parameters that detect whether or not intervals overlap or are disjoint. Trivial indiscernibility allows us to do this for several intervals at once, obtaining SOP_3 by compactness.

Conclusion

- A corollary: Every binary NSOP_3 theory is $\mathrm{NSOP}_2 = \mathrm{NTP}_1$. Every NSOP_1 binary theory is simple.
- Modulo Mutchnik's Theorem (on arXiv *today*), this would mean every NSOP_3 binary theory is simple.

Thanks!